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DYNAMICAL SUPERSYMMETRY

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Abstract

We show, in a simple quantum mechanical model, how a theory can become supersymmetric in the presence of interactions even when the free theory is not. This dynamical generation of supersymmetry relaxes the condition on the equality of masses of the superpartners which would be of phenomenological interest.

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Supersymmetry [1] is a rich theoretical concept which relates bosons to fermions. It has found applications in many diverse areas of physics [2-4]. It is also of great interest in phenomenological studies in high energy physics [5]. Conventionally, in the study of supersymmetric theories, one starts with a free theory which is invariant under supersymmetry transformations (transformations which take bosons into fermions and vice versa). This requires bosons and fermions (the superpartners) of the theory to have equal masses (or frequencies if one is dealing with quantum mechanical oscillators). Interactions are then introduced so as to maintain the tree level supersymmetry or build on it. Namely, the supersymmetry transformations of the interacting theory (if they are different from the tree level transformations) reduce to the tree level ones when interactions are switched off.

Supersymmetric theories have many interesting properties and that is, of course, the main reason for all the interest in such theories. However, the equality of masses for the superpartners is a worrisome feature of these theories since bosons and fermions with degenerate masses are not observed in nature. Spontaneous breaking of supersymmetry, on the other hand, is technically nontrivial compared to the breaking of ordinary symmetries which complicates the study of supersymmetric phenomenology. It will, therefore, be of great help if, somehow, the condition of equality of masses can be relaxed in supersymmetric theories. In this letter, we will show within the context of a quantum mechanical model how this can be achieved. More specifically, we will start with a free theory of a bosonic and a fermionic oscillator of unequal frequencies which is not supersymmetric and show that in the presence of interactions this theory can become supersymmetric. This is what we call dynamical supersymmetry and it does not require the boson and the fermion to have equal frequencies (masses).

Let us start with a quantum mechanical theory of a free bosonic and fermionic oscillator described by the Hamiltonian,

$$H_0 = \omega a^\dagger a + \epsilon c^\dagger c \tag{1}$$

where a and c stand for the bosonic and the fermionic annihilation operator respectively with ω and ϵ representing their respective frequencies. The creation and the annihilation operators for the bosons (fermions) satisfy the standard (anti) commutation relations[6].

As is well known, when $\omega = \epsilon$, this defines the supersymmetric oscillator [6-8] which is invariant under the supersymmetric transformations generated by the supercharges [6]

$$Q = a^\dagger c \quad \text{and} \quad \overline{Q} = c^\dagger a \quad (2)$$

In our entire discussion, however, we will assume that $\omega \neq \epsilon$. Our starting theory is, therefore, not supersymmetric since the bosonic and the fermionic frequencies (masses) are not equal. However, let us now look at the following interacting Hamiltonian [9-10],

$$H = \omega a^\dagger a + \epsilon c^\dagger c + g(a^\dagger + a)c^\dagger c \quad (3)$$

where g represents the strength of the interaction. We will now show that for the specific value of the coupling parameter (We assume $\epsilon > \omega$.)

$$\epsilon - \omega = \frac{g^2}{\omega} \quad (4)$$

the Hamiltonian in Eq.(3) becomes supersymmetric.

To show this, let us consider the fermionic charges

$$\begin{aligned} Q &= a^\dagger c \exp\left(\frac{g}{\omega}(a^\dagger - a)\right) \\ \overline{Q} &= \exp\left(-\frac{g}{\omega}(a^\dagger - a)\right) c^\dagger a \end{aligned} \quad (5)$$

With the standard (anti) commutation relations of the theory, it is straightforward to show that

$$\begin{aligned} [Q, H] &= \left(\epsilon - \omega - \frac{g^2}{\omega}\right) Q \\ [\overline{Q}, H] &= -\left(\epsilon - \omega - \frac{g^2}{\omega}\right) \overline{Q} \end{aligned} \quad (6)$$

It is clear now that when the condition in Eq.(4) holds, these fermionic charges are conserved and define supersymmetric transformations under which the interacting Hamiltonian in Eq.(3) is invariant. It is also straightforward to show that

$$[Q, \overline{Q}]_+ = \frac{1}{\omega} [H - (\epsilon - \omega - \frac{g^2}{\omega}) c^\dagger c] \quad (7)$$

This shows that the conserved charges Q and \overline{Q} satisfy the conventional supersymmetry algebra when Eq.(4) holds. The Hamiltonian H of Eq.(3) can be easily checked (with the condition in Eq.(4)) to be invariant under the supersymmetry transformations

$$\begin{aligned}
\delta a &= -\lambda(1 + \frac{g}{\omega}a^\dagger) \exp(\frac{g}{\omega}(a^\dagger - a))c \\
\delta a^\dagger &= -\frac{g}{\omega}\lambda a^\dagger c \exp(\frac{g}{\omega}(a^\dagger - a)) \\
\delta c &= 0 \\
\delta c^\dagger &= \lambda a^\dagger \exp(\frac{g}{\omega}(a^\dagger - a))
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\bar{\delta} a &= \frac{g}{\omega}\bar{\lambda} \exp(-\frac{g}{\omega}(a^\dagger - a))c^\dagger a \\
\bar{\delta} a^\dagger &= \bar{\lambda} \exp(-\frac{g}{\omega}(a^\dagger - a))c^\dagger(1 + \frac{g}{\omega}a) \\
\bar{\delta} c &= \bar{\lambda} \exp(-\frac{g}{\omega}(a^\dagger - a))a \\
\bar{\delta} c^\dagger &= 0
\end{aligned} \tag{9}$$

Here λ and $\bar{\lambda}$ are the two constant Grassmann parameters of the supersymmetry transformations.

Thus, we see that even though the starting theory is not supersymmetric and the bosonic and the fermionic oscillators have different frequencies (masses), for a particular value of the interaction strength, the interacting Hamiltonian has become supersymmetric. The theory has generated supersymmetry dynamically. Since the bosons and the fermions correspond to different frequencies, it is worth investigating the structure of the supersymmetric spectrum of states in this theory. It can be easily checked that the superpartner states now involve coherent states in a nontrivial way (Eq.(4) is assumed.).

$$\begin{aligned}
Q|n_a, n_c = 1\rangle &= \frac{1}{\sqrt{n_a!}}a^\dagger(a^\dagger - \frac{g}{\omega})^{n_a}|\frac{g}{\omega}, n_c = 0\rangle \\
\bar{Q}|n_a + 1, n_c = 0\rangle &= \sqrt{\frac{n_a + 1}{n_a!}}(a^\dagger + \frac{g}{\omega})^{n_a}|-\frac{g}{\omega}, n_c = 1\rangle
\end{aligned} \tag{10}$$

Here we have introduced the coherent states defined by [11]

$$|\alpha, n_c\rangle = \exp(\alpha(a^\dagger - a))|n_a = 0, n_c\rangle \tag{11}$$

Thus, we see that the relation between the perturbative supersymmetric partner states, in this case, are not as simple as in the conventional supersymmetric theories.

Finally, let us note here that this theory can be exactly solved and all of the above features can be seen in a simpler way as follows. Let us define a generalized Bogoliubov

transformation defined by the operator

$$U = \exp\left(-\frac{g}{\omega}(a^\dagger - a)c^\dagger c\right) \quad (12)$$

This defines a unitary transformation leading to

$$\begin{aligned} b &= UaU^\dagger = \left(a + \frac{g}{\omega}c^\dagger c\right) \\ b^\dagger &= Ua^\dagger U^\dagger = \left(a^\dagger + \frac{g}{\omega}c^\dagger c\right) \\ f &= UcU^\dagger = \exp\left(\frac{g}{\omega}(a^\dagger - a)\right)c \\ f^\dagger &= c^\dagger \exp\left(-\frac{g}{\omega}(a^\dagger - a)\right) \end{aligned} \quad (13)$$

These new variables satisfy the canonical (anti) commutation relations like the original fields since the transformation is unitary. It is now straightforward to check that the interacting Hamiltonian of Eq.(3) can be rewritten in terms of these new variables as

$$H = \omega b^\dagger b + \left(\epsilon - \frac{g^2}{\omega}\right)f^\dagger f \quad (14)$$

The energy eigenstates and the eigenvalues in terms of these variables are quite simple, namely,

$$H|n_b, n_f\rangle = E_{n_b, n_f}|n_b, n_f\rangle \quad (15)$$

with

$$E_{n_b, n_f} = \omega n_b + \left(\epsilon - \frac{g^2}{\omega}\right)n_f \quad (16)$$

with $n_f = 0, 1$ and $n_b = 0, 1, 2, \dots$. It is clear now that when Eq.(4) is satisfied the theory is nothing other than the supersymmetric oscillator in terms of these new variables. The supersymmetric partner states are the conventional ones in the quanta of the redefined variables. We also note that when $\epsilon - \frac{g^2}{\omega} < 0$, the ground state of the theory becomes fermionic [9] whereas if $\epsilon - \frac{g^2}{\omega} = 0$, the fermions completely drop out of the theory. Simple as the Hamiltonian in Eq.(3) may appear to be, it really has a rich structure. It is clear now (see, e.g., [6-8]) from the form of the Hamiltonian in Eq.(14) and the unitary transformation in Eq.(12) that one could also have started with a more complicated interacting Hamiltonian in Eq.(3) which would have resulted in a supersymmetric, interacting Hamiltonian in terms of the variables b and f .

To conclude, we have shown in a simple quantum mechanical model how supersymmetry can be dynamically generated in the presence of interactions even when the free theory may not be supersymmetric. It remains to be seen if and how this idea can be generalized to relativistic quantum field theories. The properties of such theories would be quite interesting to investigate.

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